Phase transition in a random Ising model on a Cayley tree

1980 J. Phys. A: Math. Gen. 13 L71

(http://iopscience.iop.org/0305-4470/13/3/008)

J. Phys. A: Math. Gen. **13** (1980) L71–L75. Printed in Great Britain

LETTER TO THE EDITOR

Phase transition in a random Ising model on a Cayley tree

T Horiguchi and T Morita

Department of Applied Science, Faculty of Engineering, Tohoku University, Sendai 980, Japan

Received 14 November 1979

Abstract. Temperatures where the $2n$th expansion coefficient of the average free energy with respect to the external field fails to have a definite value are obtained for a random Ising model on a Cayley tree. The assumptions made for the random Ising model are that the exchange integral is a random variable, its probability distribution is independent of the pair of sites and the average value of the absolute value of the exchange integral is finite. The critical exponents κ defined by Müller-Hartmann and Zittartz are obtained. The phase transition is of the continuous order.

The regular Ising model on a Cayley tree has been shown to exhibit an unusual phase transition by many authors (Eggarter 1974, Matsuda 1974, Müller-Hartmann and Zittartz 1974, von Heimburg and Thomas 1974, Morita and Horiguchi 1975, Falk 1975). In particular, Müller-Hartmann and Zittartz (1974, 1975) revealed that the phase transition is of the continuous order. Attention has been extended to the random Ising model. Heinrichs (1979) has investigated a dilute Ising model in the vicinity of the percolation threshold at low temperatures. Gonçalves da Silva (1979) has studied the random Ising model with an equal number of ferro- and antiferromagnetic bonds.

In this Letter, we report our results obtained recently for a random Ising model on a Cayley tree in which the probability distribution of the random exchange integral is assumed to be independent of the pair of sites.

Consider the Cayley tree built up of $N+1$ generations. The zeroth generation has only one site, which is denoted by 0 or r_0 and has B nearest-neighbour sites belonging to the first generation. Each site of the nth generation, except $n=0$ and $n=N$, has $B+1$ nearest-neighbour sites of which B belong to the $(n+1)$th generation and one to the $(n-1)$th generation. Each site of the Nth generation has only one nearest-neighbour site belonging to the $(N-1)$th generation. A site of the nth generation $(n \neq 0)$ is labelled by $i_1 i_2 \ldots i_n$, where i_l are integers between 1 and B, or briefly by r_n. The total number of sites N_s is given by

$$N_s = (B^{N+1} - 1)/(B-1). \qquad (1)$$

Our system is described by the Hamiltonian

$$-\beta H_N = C\sigma_0 + \sum_{i_1=1}^{B} \left\{ K_{i_1}\sigma_0\sigma_{i_1} + C\sigma_{i_1} + \sum_{i_2=1}^{B} \left[K_{i_1 i_2}\sigma_{i_1}\sigma_{i_1 i_2} + C\sigma_{i_1 i_2} \right.\right.$$
$$\left.\left. + \ldots + \sum_{i_N=1}^{B} (K_{i_1 i_2 \ldots i_N}\sigma_{i_1 i_2 \ldots i_{N-1}}\sigma_{i_1 i_2 \ldots i_N} + C\sigma_{i_1 i_2 \ldots i_N}) \ldots \right] \right\}, \qquad (2)$$

0305-4470/80/030071+05\$01.00

where σ_{r_n} is the Ising spin variable and $K_{r_n} = \beta J_{r_n}$, $C = \beta h$, and $\beta = 1/k_B T$ as usual. h is an applied external field and J_{r_n} is a random exchange integral whose probability density or mass $P(J_{r_n})$ is assumed not to depend on each pair of sites.

Under the condition that the average of the absolute value of the random variable J_{r_n} is finite, we are able to show that the average free energy per site $f(h)$ in the thermodynamic limit is expressed by

$$-\beta f(h) = \frac{B-1}{2} \sum_{\nu=1}^{\infty} B^{-\nu} \int \left\{ \int \ln[2\cosh(2y) + 2\cosh(2K)] P(J)\, dJ \right\} g^{(\nu-1)}(y)\, dy, \tag{3}$$

where $K = \beta J$,

$$g^{(\nu)}(y) = \int \dots \int \left\{ \int \dots \int \delta\left[y - C - \sum_{i=1}^{B} \tanh^{-1}(\tanh y_i \tanh K_i) \right] \right.$$
$$\left. \times \prod_{i=1}^{B} [P(J_i)\, dJ_i] \right\} \prod_{i=1}^{B} [g^{(\nu-1)}(y_i)\, dy_i] \tag{4}$$

for $\nu = 1, 2, \dots$, and

$$g^{(0)}(y) = \delta(y - C). \tag{5}$$

The right-hand side of (3) converges uniformly. An expression for the average magnetisation per site in the thermodynamic limit is also obtained and the uniformity of convergence of the obtained expression gives the conclusion that there is no spontaneous magnetisation except at $T = 0$ in our system.

Consider the free energy, which is the one obtained by subtracting $-\beta f(0)$ from (3):

$$-\beta \Delta f(h) = \frac{B-1}{2} \sum_{\nu=1}^{\infty} B^{-\nu} \int \left[\int \ln\left(1 + \frac{\sinh^2 y}{\cosh^2 K}\right) P(J)\, dJ \right] g^{(\nu-1)}(y)\, dy. \tag{6}$$

If $-\beta \Delta f(h)$ has the first $2n$ derivatives at $h = 0$, then we have

$$-\beta \Delta f(h) = \sum_{l=1}^{n} f_l C^{2l} + o(C^{2n}), \tag{7}$$

where $o(C^{2n})$ expresses a term of order less than C^{2n} (e.g. Hardy 1952). Here we have

$$f_l = \frac{B-1}{2} \sum_{\nu=1}^{\infty} B^{-\nu} \sum_{n=0}^{l-1} \sum_{p=1}^{n+1} \frac{(-1)^{p-1}}{p} \left(\int (1 - \tanh^2 K)^p P(J)\, dJ \right)$$
$$\times F(1 + n - p, 2p, \{1/(2j+1)!\}) \tilde{y}_{\nu-1}^{(2+2n, 2l)}, \tag{8}$$

where

$$F(l, m, \{f_j\}) = \sum_{\{m_j\}} \frac{m!}{\prod_j m_j!} \prod_j [f_j]^{m_j}. \tag{9}$$

The summation on the right-hand side of (9) is taken over the sets $\{m_j\}$ of non-negative integers m_j, which satisfy the conditions $\sum_{j=0}^{m} m_j = m$ and $\sum_{j=0}^{m} j m_j = l$. $\tilde{y}_\nu^{(2+2n, 2l)}$ is determined from the recursion formula

$$\tilde{y}_{\nu+1}^{(m, m+2l)} = \sum_{k=0}^{m} \binom{m}{k} \sum_{\{p_i, l_i\}} \frac{k!B!}{\prod_i [(p_i!)^{l_i} l_i!](\sum_i l_i)!} \sum_{\{\omega_i\}} \prod_{i=1}$$
$$\times F\left(\omega_i, l_i, \left\{ \sum_{j_1=1}^{j} Q_{j_1}(p_i) \tilde{y}_\nu^{(p_i + 2j_1, p_i + 2j)} \right\} \right), \tag{10}$$

$$Q_{j_1}(p_i) = \sum_{j_2=0}^{j_1} F\left(j_2, p_i, \left\{\frac{1}{2j+1}\right\}\right) F\left(j_1 - j_2, p_i + 2j_2, \left\{\frac{(-1)^j T_{j+1}}{(2j+1)!}\right\}\right)$$

$$\times \int (\tanh K)^{p_i + 2j_2} P(J)\, \mathrm{d}J, \tag{11}$$

$$\tilde{y}_0^{(m,m)} = 1, \qquad \tilde{y}_0^{(m,m+2l)} = 0 \qquad (l \neq 0); \tag{12}$$

T_j are defined by $2^{2j}(2^{2j}-1)B_j/2j$ where B_j are the Bernoulli numbers. The second summation on the right-hand side of (10) is taken over the sets $\{p_i, l_i\}$ of positive integers $p_1, p_2, \ldots, p_n, l_1, l_2, \ldots, l_n$, which satisfy the conditions $\Sigma_{i=1}^n l_i p_i = k$ and $p_{i+1} > p_i$, and the third summation is taken over the sets $\{\omega_i\}$ of non-negative integers ω_i, satisfying the condition $\sum_{i=1}^{l} \omega_i = l$.

We now find the highest temperature $k_B T^{(2n)} (= 1/\beta^{(2n)})$ at which f_n fails to have a definite value. Taking into account the inequality

$$\left| \int \tanh^l K\, P(J) \mathrm{d}J \right| \leq \int \tanh^l |K| P(J) \mathrm{d}J \leq \int \tanh^2 K\, P(J) \mathrm{d}J \tag{13}$$

for $l = 3, 4, \ldots$, we conclude that the sum of the most dangerous terms in f_n is

$$f_n \sim \alpha_n \sum_{\nu=1}^{\infty} B^{-\nu} \left[B \int \tanh K\, P(J) \mathrm{d}J \right]^{2n\nu} + \beta_n \sum_{\nu=1}^{\infty} B^{-\nu} \left[B \int \tanh^2 K\, P(J) \mathrm{d}J \right]^{n\nu}. \tag{14}$$

Then we have two possible temperatures for $T^{(2n)}$ which are obtained from the following equations, respectively:

$$\left| \int \tanh K\, P(J) \mathrm{d}J \right| = B^{(1/2n)-1}, \tag{15}$$

$$\int \tanh^2 K\, P(J) \mathrm{d}J = B^{(1/n)-1}. \tag{16}$$

If we denote the positive solution β of $\int \tanh^2 K\, P(J) \mathrm{d}J = x$ for $0 \leq x \leq 1$ by $\beta_2(x)$, then the solution of (16), $\beta_2^{(2n)}$, is given by

$$\beta_2^{(2n)} = \beta_2(B^{(1/n)-1}). \tag{17}$$

Here we note that we always have a non-zero solution $k_B T_2^{(2n)} (= 1/\beta_2^{(2n)})$ except for $n = 1$. If we denote the smallest positive solution β of $|\int \tanh K\, P(J) \mathrm{d}J| = x$ for $0 \leq x \leq 1$ by $\beta_1(x)$, the solution of (15), $\beta_1^{(2n)}$, is given by

$$\beta_1^{(2n)} = \beta_1(B^{(1/2n)-1}). \tag{18}$$

We note that we do not always have the solution $\beta_1^{(2n)}$. When there is no solution $\beta_1^{(2n)}$, we put $\beta_1^{(2n)} = \infty$. Then we have

$$\beta^{(2n)} = \min(\beta_1^{(2n)}, B_2^{(2n)}). \tag{19}$$

Noticing that each of $\beta_1(x)$ and $\beta_2(x)$ is a monotonic function of x, we have

$$T^{(2)} \leq T^{(4)} \leq \ldots \leq T^{(\infty)}, \tag{20}$$

where $T^{(\infty)}$ is the highest temperature which satisfies either $B|\int \tanh K\, P(J) \mathrm{d}J| = 1$ or $B \int \tanh^2 K\, P(J) \mathrm{d}J = 1$. In a separate paper, we give an expression for the '$2n$th order susceptibility' f_n at high temperatures in the thermodynamic limit. The expression involves terms which diverge at the temperature $T^{(2n)}$ given by (19).

For the system with ferro- and antiferromagnetic bonds of magnitude $|J|>0$, i.e. when

$$P(J_{r_n}) = p\delta(J_{r_n} - |J|) + (1-p)\delta(J_{r_n} + |J|), \tag{21}$$

we show $T^{(2n)}$ given by (19) in figure 1 for $B=2$.

Finally, by using the cut-off argument by Müller-Hartmann and Zittartz, we have the critical exponents $\kappa(T)$ for the leading non-analytic behaviour of the form $|h|^{\kappa(T)}$ of the free energy (6) as follows:

$$\kappa^{(1)}(T) = \ln B/\ln\left(B\left|\int \tanh K\, P(J)\mathrm{d}J\right|\right), \tag{22}$$

$$\kappa^{(2)}(T) = 2\ln B/\ln\left(B\int \tanh^2 K\, P(J)\mathrm{d}J\right). \tag{23}$$

Thus the phase transition is of the continuous order κ.

In concluding this Letter, we remark that the results obtained by specialising our results to the model with (21) are different from the ones by Gonçalves da Silva (1979) except at $p=\frac{1}{2}$. He studied the ferromagnetic Ising model with random applied field, which is not equivalent to the random Ising model with ferro- and antiferromagnetic bonds except at $p=\frac{1}{2}$. The details of the present calculations and further discussions will be presented elsewhere.

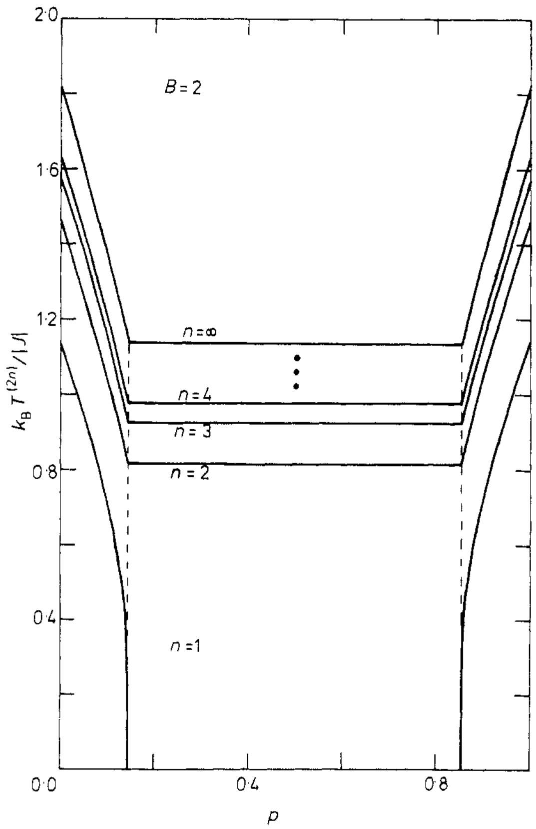

Figure 1. $k_B T^{(2n)}/|J|$ versus concentration p of ferromagnetic bonds for $B=2$.

References

Eggarter T P 1974 *Phys. Rev.* B **9** 2989–92
Falk H 1975 *Phys. Rev.* B **12** 5184–9
Gonçalves da Silva C E T 1979 *J. Phys. C: Solid St. Phys.* **12** L219–22
Hardy G H 1952 *A Course of Pure Mathematics* (10th ed Cambridge) p 290
Heinrichs J 1979 *Phys. Rev.* B **19** 3788–98
Matsuda H 1974 *Prog. Theor. Phys.* **51** 1053–63
Morita T and Horiguchi T 1975 *Prog. Theor. Phys.* **54** 982–98
Müller-Hartmann E and Zittartz J 1974 *Phys. Rev. Lett.* **33** 893–7
—— 1975 *Z. Phys.* B **22** 59–67
von Heimburg J and Thomas H 1974 *J. Phys. C: Solid St. Phys.* **7** 3433–43